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Integrals of motion and analytic functions

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Abstract. In autonomous dynamical systems with two degrees of freedom, any integral polynomial in the velocities determines uniquely a complex analytic function. Conversely, given any analytic function there exists a polynomial integral of a dynamical system which determines this function. The analytic function provides an immediately verifiable necessary criterion which can answer the question of whether an expression which is polynomial in the velocities can be an integral of any (unknown) two-dimensional dynamical system. The basic operation among integrals corresponds to similar operations among the corresponding analytic functions.

1. Introduction

There is still a great deal of interest in the 'inverse problem' in mechanics, namely, the determination of the forces which act on a dynamical system from the knowledge of several characteristics of the motions of the system. Usually the task is to determine the potential which generates a given family of orbits (Szebehely 1974, Broucke and Lass 1977).

Another version of the 'inverse problem' is the following. Given that a conservative system admits an integral (of motion) of some specified form, determine the forces acting on the system (equivalently, determine the potential) and the integral explicitly. For two degrees of freedom and integrals quadratic in the velocities (equivalently, in the momenta) the solution of the problem is given by Whittaker (1937) (due originally to Bertran). Chandrasekhar (1960) has solved the same problem for integrals quadratic in the velocities in systems with three degrees of freedom.

It seems unjustified, however, to restrict considerations to integrals quadratic in the velocities. For instance, studies in celestial mechanics (Contopoulos 1965, Bozis 1966) and galactic dynamics (Contopoulos 1960, 1979, Barbanis 1962) have indicated the need to use integrals which are of the fourth power in the velocities. For instance, Bozis (1982) has recently addressed the inverse problem by considering integrals of the fourth power in the velocities in systems with two degrees of freedom and determining the potential which admits this integral as well as the explicit form of the integral.

In this paper we study the existence of the integrals of motion in autonomous dynamical systems (i.e. conservative systems with time independent potentials), with two degrees of freedom, which (integrals) are polynomial in the velocities. A necessary condition is obtained for an expression polynomial in the velocities to be an integral; the condition involves the coefficients of the polynomial but not the potential itself. Therefore, it can be used as a necessary criterion to check immediately whether an

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algebraic expression can be an integral of motion in some unknown potential. The necessary condition is that a simple linear combination of the coefficients of the polynomial—given by equation (2.15)—is a complex analytic function in the cartesian (x, y) plane. In § 3 we show that, given any complex analytic function and any positive integer n, there exists a potential and a polynomial homogeneous in the velocities of degree n which is an integral of the constant energy motions of the potential and whose corresponding analytic function is the given one. The theory (which associates integrals of motion with analytic functions), developed in §§ 2 and 3 for polynomial integrals homogeneous in the velocities, is extended to any integral which is polynomial in the velocities in § 4. Finally, in § 5 we determine the complex analytic function which corresponds to the sum, the product, and the Poisson bracket of two polynomial integrals.

2. The analytic function

We consider an autonomous dynamical system with two degrees of freedom, potential $V = V(x, y)$ which is a C^1 function in the cartesian (x, y) plane, and equations of motion

$$\ddot{x} = -V_{,x}, \qquad \ddot{y} = -V_{,y} \tag{2.1}$$

admitting the energy integral

$$e = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + V(x, y), \tag{2.2}$$

where the dot denotes differentiation with respect to time t. In addition we assume that the homogeneous polynomial in the velocities of degree n

$$I = \sum_{p=0}^{n} A_p^n(x, y)\dot{x}^p\dot{y}^{n-p} \tag{2.3}$$

is an integral of equations (2.1). To simplify the subsequent notation we adopt the convention

$$A_p^n = 0 \qquad \text{for } p < 0 \text{ and for } p > n \tag{2.4}$$

for the coefficients of the integral.

The condition $\mathrm{d}I/\mathrm{d}t = 0$ that the polynomial (2.3) is an integral reads

$$\sum_{p=0}^{n} [A_{p,x}^n \dot{x}^{p+1}\dot{y}^{n-p} + A_{p,y}^n \dot{x}^p\dot{y}^{n-p+1} - pA_p^n\dot{x}^{p-1}\dot{y}^{n-p}V_{,x} - (n-p)A_p^n\dot{x}^p\dot{y}^{n-p-1}V_{,y}] = 0. \tag{2.5}$$

To express this condition homogeneously in the velocities we multiply the last two terms by

$$(\dot{x}^2 + \dot{y}^2)/2(e - V) = 1, \tag{2.6}$$

which is the energy integral (2.2). We obtain that

$$\sum_{p=0}^{n} [A_{p,x}^n \dot{x}^{p+1}\dot{y}^{n-p} + A_{p,y}^n \dot{x}^p\dot{y}^{n-p+1} + pA_p^n\dot{x}^{p+1}\dot{y}^{n-p}X + pA_p^n\dot{x}^{p-1}\dot{y}^{n-p+2}X + (n-p)A_p^n\dot{x}^{p+2}\dot{y}^{n-p-1}Y + (n-p)A_p^n\dot{x}^p\dot{y}^{n-p+1}Y] = 0 \tag{2.7}$$

where we have introduced the notation

$$\begin{pmatrix} X \\ Y \end{pmatrix} = -\frac{1}{2(e-V)}\begin{pmatrix} V_{,x} \\ V_{,y} \end{pmatrix}. \tag{2.8}$$

Finally, by suitably changing the summation indices to convert all the velocity terms to the form $\dot{x}^p\dot{y}^{n-p+1}$, using the conventions (2.4) and equating to zero the coefficients of the resulting expressions which are polynomial in the velocities we obtain the necessary and sufficient conditions for the polynomial (2.3) to be an integral. They are the following $n+2$ equations:

$$\begin{aligned} &A^n_{p-1,x} + A^n_{p,y} + (p-1)A^n_{p-1}X + (p+1)A^n_{p+1}X + (n-p+2)A^n_{p-2}Y \\ &\qquad + (n-p)A^n_p Y = 0, \qquad p = 0, 1, \ldots, n+1. \end{aligned} \tag{2.9}$$

By introducing the notation

$$Y^n_p = pA^n_p X + (n+1-p)A^n_{p-1}Y \tag{2.10}$$

these equations read

$$A^n_{p-1,x} + A^n_{p,y} + Y^n_{p-1} + Y^n_{p+1} = 0, \qquad p = 0, 1, \ldots, n+1. \tag{2.11}$$

Since the unknown potential appears only in the Y^n_p's our aim is to eliminate the quantities Y^n_p from equations (2.11).

We consider first those of equations (2.11) corresponding to the even values of the index p. By multiplying each equation by $(-1)^{p/2}$, adding the resulting equations and using $Y^n_0 = Y^n_{n+1} = 0$ we obtain

$$P_{,y} + Q_{,x} = 0, \tag{2.12}$$

where

$$\begin{aligned} P &= \sum_{p=\text{even}}^{0,n+1} (-1)^{p/2} A^n_p = \sum_{p=\text{odd}}^{0,n+1} (-1)^{(p-1)/2} A^n_{p-1}, \\ Q &= \sum_{p=\text{even}}^{0,n+1} (-1)^{p/2} A^n_{p-1} = -\sum_{p=\text{odd}}^{0,n+1} (-1)^{(p-1)/2} A^n_p. \end{aligned} \tag{2.13}$$

Similarly, by considering those of equations (2.11) corresponding to odd values of the index p, multiplying them by $(-1)^{(p-1)/2}$ and summing the resulting equations we obtain

$$P_{,x} - Q_{,y} = 0. \tag{2.14}$$

We conclude, therefore, that the functions P and Q, which are determined solely from the coefficients of the integral (2.3), are conjugate harmonic functions. Equivalently, the function $f = P + \mathrm{i}Q$, which is easily determined from the coefficients of the integral by

$$f(z) = f(x, y) = \sum_{p=0}^{n} (-\mathrm{i})^p A^n_p(x, y), \tag{2.15}$$

should be an analytic function of one complex variable. We shall call the function (2.15) the analytic function of the integral (2.3).

As an example we determine the analytic function of the energy integral. We write it in a form homogeneous in the velocities $e = e(\dot{x}^2 + \dot{y}^2)/2(e - V)$ by multiplying the potential energy V by the identity (2.6). Therefore, $A_0^2 = A_2^2$ and $A_1^2 = 0$ and its analytic function is $f(z) = 0$. Similarly we find that the analytic function of the integral of the angular momentum $I = x\dot{y} - y\dot{x}$ (whenever it is conserved) is $f(z) = z$.

3. The converse

We have established in § 2 that to any homogeneous polynomial integral there corresponds a complex analytic function. In this section we show the converse of the above statement. Precisely, we prove the following.

Theorem 1. Given any complex analytic function f and any positive integer n there exists a potential $V = V(x, y)$ of an autonomous dynamical system with two degrees of freedom and a polynomial of degree n homogeneous in the velocities which is an integral of the constant energy motions of the system and for which the corresponding analytic function is f.

For the proof we have to show that for any analytic function f and any positive integer n there exists a solution of the system of equations (2.8), (2.10) and (2.11) with constant e, such that the analytic function determined from equation (2.15) is the given function f. We have found a simple solution of the system of these equations. The solution is

$$A_p^n = 2^{-n}\, \mathrm{i}^p \binom{n}{p} [f + (-1)^p \bar{f}], \qquad p = 0, 1, \ldots, n, \tag{3.1}$$

$$\ln (e - V) = -n^{-1}(\ln f + \ln \bar{f}). \tag{3.2}$$

It is straightforward to verify that

$$\begin{aligned} Y_p^n &= -2^{-n}\, \mathrm{i}^p \binom{n-1}{p-1} [f_{,z} + (-1)^p \bar{f}_{,\bar{z}}], \\ A_{p-1,x}^n &= 2^{-n}\, \mathrm{i}^{p-1} \binom{n}{p-1} [f_{,z} + (-1)^{p-1} \bar{f}_{,\bar{z}}], \\ A_{p,y}^n &= 2^{-n}\, \mathrm{i}^{p+1} \binom{n}{p} [f_{,z} + (-1)^{p-1} \bar{f}_{,\bar{z}}], \end{aligned} \tag{3.3}$$

and then observe that equations (2.11) are indeed satisfied. Moreover, by using $\Sigma_p \binom{n}{p} = 2^n$ and $\Sigma_p (-1)^p \binom{n}{p} = 0$ it is easy to see that the condition (2.15) is also satisfied. Therefore, the existence of the solution (3.1) and (3.2) shows the validity of theorem 1.

We close this section with the following three remarks.

(i) The coefficients (3.1) of the integral are all real.

(ii) Equation (3.2) implies that the potential is $V = e - (\bar{f})^{-1/n}$, i.e. that it is a central potential.

(iii) The corresponding integral is easily found to be

$$I = 2^{-n}\, i^n [f(z)\dot{\bar{z}}^n + (-1)^n \bar{f}(\bar{z})\dot{z}^n], \tag{3.4}$$

where $z = x + iy$. Obviously, I is real.

4. Polynomial integrals

Here we extend the results of § 2 to integrals of autonomous systems which are polynomial in the velocities, but are *not necessarily homogeneous.* Obviously, if the polynomial integral contains terms which are even and terms which are odd in the velocities, the part of the integral consisting of all the terms even in the velocities is conserved independently of the part consisting of all the odd terms. Without any loss of generality, therefore, we will consider only integrals which are polynomial in the velocities and of definite parity.

Any integral polynomial in the velocities of definite parity can be written in an equivalent form in which the integral is a polynomial homogeneous in the velocities. This can be done by using the energy integral in the form (2.6) and multiplying the lower degree terms by suitable powers of $\alpha(\dot{x}^2 + \dot{y}^2) = 1$, where $\alpha^{-1} = 2(e - V)$. For instance, the polynomial even (non-homogeneous) in the velocities

$$I = \sum_{s=0}^{m} \sum_{p=0}^{2s} A_p^{2s} \dot{x}^p \dot{y}^{2s-p} \tag{4.1}$$

can be written, equivalently, in the homogeneous form

$$I = \sum_{s=0}^{m} \sum_{p=0}^{2s} A_p^{2s} \alpha^{m-s} (\dot{x}^2 + \dot{y}^2)^{m-s} \dot{x}^p \dot{y}^{2s-p}. \tag{4.2}$$

At first glance it appears, however, that the necessary condition established in § 2 for a homogeneous polynomial to be an integral cannot be applied to the general polynomial, of the form (4.1). Although we can still evaluate its complex function f—which must be analytic—by using the equivalent homogeneous expression (4.2) of the integral, the resulting function f is expected to depend on the unknown potential as well—via α—and therefore its analyticity cannot be verified. This impression, however, is superficial. The instructions for constructing the analytic function f of an integral which is a homogeneous polynomial are 'perform the substitutions $\dot{x} = -i$ and $\dot{y} = 1$'.

For the integral (4.1) this substitution should be performed in its equivalent form (4.2). Since $\dot{x}^2 + \dot{y}^2 = (-i)^2 + 1^2 = 0$, the corresponding analytic function is

$$f = \sum_{p=0}^{2m} (-i)^p A_p^{2m}, \tag{4.3}$$

a function independent of the potential V, whose analyticity can be immediately verified. We have established, therefore, the following.

Theorem 2. A necessary condition for a polynomial (in the velocities) expression of definite parity to be an integral of an autonomous dynamical system with two degrees of freedom is that the function f obtained by considering the terms of higher degree in the velocities and setting $\dot{x}=-\mathrm{i}$ and $\dot{y}=1$ is a complex analytic function in the cartesian (x, y) plane.

5. Properties of the analytic function

In this section we show that to the three basic operations between integrals—addition, multiplication, and Poisson bracket skew multiplication—there correspond three similar operations between the analytic functions of the integrals.

Theorem 3. The analytic function of the sum of two integrals which are polynomial in the velocities of definite parity and of the same degree equals the sum of the analytic functions of the two integrals.

The proof follows immediately from the linearity of the expression (2.15) in the coefficients of the integral.

Theorem 4. The analytic function of the product of two integrals which are polynomial in the velocities equals the product of the analytic functions of the two integrals.

Obviously, we only have to show the theorem for polynomials homogeneous in the velocities. Let

$$I_m=\sum_p A_p^m \dot{x}^p \dot{y}^{m-p}, \qquad I_n=\sum_q A_q^n \dot{x}^q \dot{y}^{n-q} \tag{5.1}$$

be two homogeneous integrals of degrees m and n and let

$$f_m=\sum_p (-\mathrm{i})^p A_p^m, \qquad f_n=\sum_q (-\mathrm{i})^q A_q^n \tag{5.2}$$

be their analytic functions, respectively. The coefficients of the integral $I_{m+n}=I_m I_n$ are $A_s^{m+n}=\Sigma_{p+q=s} A_p^m A_q^n$ and therefore the corresponding analytic function is

$$f_{m+n}=\sum_s (-\mathrm{i})^{p+q} \sum_{p+q=s} A_p^m A_q^n=\sum_p (-\mathrm{i})^p A_p^m \times \sum_q (-\mathrm{i})^q A_q^n f_m f_n.$$

Theorem 5. The analytic function of the Poisson bracket $[I_m, I_n]$ of two integrals which are polynomial in the velocities of degrees m and n, respectively, is given by

$$f_{m+n-1}=\mathrm{i}\,(n f_n\, \mathrm{d}f_m/\mathrm{d}z-m f_m\, \mathrm{d}f_n/\mathrm{d}z), \tag{5.3}$$

where f_m and f_n are the analytic functions of the two integrals I_m and I_n.

Obviously, again we have to show the theorem only for integrals which are homogeneous polynomials in the velocities, say the integrals (5.1). Then the Poisson bracket is an integral which is a homogeneous polynomial of degree $m+n-1$. We label f_{m+n-1} the analytic function of the Poisson bracket.

By using the expressions (5.1) and performing the required differentiations we obtain for the Poisson bracket the expression

$$[I_m, I_n] = \sum_{p,q} (qA^m_{p,x}A^n_q - pA^m_p A^n_{q,x})\dot{x}^{p+q-1}\dot{y}^{m+n-p-q} + \sum_{p,q} [(n-q)A^m_{p,y}A^n_q - (m-p)A^m_p A^n_{q,y}]\dot{x}^{p+q}\dot{y}^{m+n-p-q-1} \tag{5.4}$$

which, by setting $p+q=k+1$ in the first and $p+q=k$ in the second summation becomes

$$[I_m, I_n] = \sum_{k=0}^{m+n-1} \sum_{p+q=k+1} (qA^m_{p,x}A^n_q - pA^m_p A^n_{q,x})\dot{x}^k\dot{y}^{m+n-k-1} + \sum_{k=0}^{m+n-1} \sum_{p+q=k} [(n-q)A^m_{p,y}A^n_q - (m-p)A^m_p A^n_{q,y}]\dot{x}^k\dot{y}^{m+n-k-1}. \tag{5.5}$$

By further changing p into $p+1$ in the first term of the first summation and q into $q+1$ in the second term of the first summation and regrouping the terms, we can write the Poisson bracket in the form

$$[I_m, I_n] = \sum_{k=0}^{m+n-1} A^{m+n-1}_k \dot{x}^k \dot{y}^{m+n-k-1} \tag{5.6}$$

where

$$A^{m+n-1}_k = \sum_{p+q=k} [qA^m_{p+1,x}A^n_q - pA^m_p A^n_{q+1,x} + (n-q)A^m_{p,y}A^n_q - (m-p)A^m_p A^n_{q,y}]. \tag{5.7}$$

It is now straightforward to write the corresponding analytic function. We obtain

$$\begin{aligned} f_{m+n-1} &= \sum_{p=0}^{m} \sum_{q=0}^{n} [q(-\mathrm{i})^p(-\mathrm{i})^q A^m_{p+1,x}A^n_q - p(-\mathrm{i})^p(-\mathrm{i})^q A^m_p A^n_{q+1,x} \\ &\quad + (n-q)(-\mathrm{i})^p(-\mathrm{i})^q A^n_{p,y}A^n_q - (m-p)(-\mathrm{i})^p(-\mathrm{i})^q A^m_p A^n_{q,y}] \\ &= \sum_q \mathrm{i}q(-\mathrm{i})^q A^n_q \,\mathrm{d}f_m/\mathrm{d}z - \sum_p \mathrm{i}p(-\mathrm{i})^p A^m_p \,\mathrm{d}f_n/\mathrm{d}z \\ &\quad + \sum_q \mathrm{i}(n-q)(-\mathrm{i})^q A^n_q \,\mathrm{d}f_m/\mathrm{d}z - \sum_p \mathrm{i}(m-p)(-\mathrm{i})^p A^m_p \,\mathrm{d}f_n/\mathrm{d}z \\ &= \sum_q \mathrm{i}n\,(\mathrm{d}f_m/\mathrm{d}z)(-\mathrm{i})^q A^n_q - \sum_p \mathrm{i}m(-\mathrm{i})^p A^m_p \,\mathrm{d}f_n/\mathrm{d}z \\ &= \mathrm{i}(nf_n \,\mathrm{d}f_m/\mathrm{d}z - mf_m \,\mathrm{d}f_n/\mathrm{d}z), \end{aligned} \tag{5.8}$$

where we have also used that $f_{,x} = \mathrm{d}f/\mathrm{d}z$ and $f_{,y} = \mathrm{i}\,\mathrm{d}f/\mathrm{d}z$ for a complex analytic function.

6. Discussion

The main conclusion of the present paper is that to any integral which is polynomial in the velocities of an autonomous dynamical system there corresponds a complex analytic function and that to the integral obtained by the addition (of polynomials of the same degree), the multiplication and by the Poisson bracket of two such integrals there correspond the sum, the product, and the function given by the expression (5.3) of the two analytic functions of the two original integrals.

The association of analytic functions with integrals can be used as a necessary criterion which immediately checks whether a given function which is polynomial in the velocities can be an integral of some unknown dynamical system. It should be mentioned, however, that the criterion does not seem to be applicable to integrals which are power series in the velocities.

The above criterion is only necessary. In principle it is possible to obtain the necessary and sufficient conditions on the coefficients A_p^n for the polynomial (2.3) to be an integral. One has to eliminate X and Y among equations (2.11) and, in addition, include the integrability conditions implied by the existence of V. This programme, however, does not seem to lead to any simple conditions on A_p^n.

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